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IMPACT AND CONTACT STRESS ANALYSIS IN MULTILAYER MEDIA*

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Abstract—The contact problem in a multilayer medium is analyzed based upon classical elasticity theory. The mixed boundary value problem is reformulated into a general approximation technique suitable for calculation on a digital computer. The Boussinesq problem demonstrates that this approximate method is much more accurate than an "exact" analysis recently published. A second example, which is most commonly applied in engineering practice, involves the parabolic punch. Numerical results are presented for both examples, to illustrate the physically significant effects of soft and hard layers and different layer thicknesses. The analysis is then extended to quasi-static impact. Results of experiments dropping a steel ball on nylon and rubber layers over a granite base are given and good agreement is found between the analytical and experimental results.

1. INTRODUCTION

THE indentation of a medium consisting of two (or one) parallel elastic layers bonded to a homogeneous half-space is the prime topic of this study.

Although the contact problems of a homogeneous half-space and of a slab have received considerable attention in the literature [1–3], and much is known concerning the first boundary value problem in multilayer media [4], relatively little is known of contact analysis of multilayer media. In modern engineering practice, one frequently meets with stratified structures, laminated composites or solid components coated with surface layers, therefore, this contact analysis is often necessary.

In this work, a general approximate method for the above problem will be developed and its capability, range and practicality will be demonstrated through the example of the Boussinesq problem. Various geometric and material parameters will be examined for their effect in Hertz type indentation problems. The analysis will then be extended to quasi-static impact, and experimental verification will be sought to some of the analytical results.

The contact problem is a mixed boundary value problem. Much of the literature in this area deals with an elastic slab, either loaded symmetrically from both surfaces or loaded from one surface and rigidly fixed on the other surface. A survey of these two types of elasticity problems, tackled by the method of integral transforms, may be found in the book by Uflyand [5]. The mixed boundary value problem is formulated as a dual integral equation and reduced into a Fredholm integral equation, which is then solved approximately [6–34]. A second approach would be to replace the exact boundary condition by an approximate one, and solve the new elasticity problem exactly [35–40].

The rigid stratum supporting the elastic slab is, of course, a mathematical fiction that simplifies the analysis. Some recent investigators have considered the indentation of an elastic layer bonded to an elastic half-space made up of a different material. Wu and

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Chiu [41] considered the plane strain problem and reduced the mixed boundary value problem to a single Fredholm integral equation of the second kind. Dhaliwal [42] was able to reduce the axisymmetric problem of a flat ended cylindrical punch on this layered elastic medium, to a Fredholm integral equation which he solved approximately. For the so-called "thick" layer situation, he developed a series solution iteratively. Subsequently Dhaliwal and Rau [43] extended the analysis of Dhaliwal to punches of arbitrary profile, but did not present any numerical results.

Engel [44] first made use of the numerical solution in the static single slab analysis to study impact on a viscoelastic slab adopting the quasi-static assumptions made by Hertz. A related approach was taken to make a force-time study of the elastic impact on a finite slab by Conway *et al.* [45] and the results were correlated with experiment.

Various methods of solving this mixed boundary problem may be considered in the context of numerical approximation of functions. First, one assumes a set of pressure base functions (preferably orthogonal and complete), which satisfies all of the equations of elasticity internally and the stress boundary condition. This set of base functions is assembled to represent the contact pressure distribution so that the displacement within the contact area approximates the real displacement boundary conditions in accordance with some particular boundary residual error criterion.

The "point matching" method employed in [36, 37, 39, 40] may be considered as one such method. Here, the assumed base function is made up of uniform pressure applied over an annulus within the contact regions, and the boundary residual criterion is that of collocation.

Similarly, the form of the base function is already assumed when the kernel of the integral equation is expanded in a power series (see equations (4.2) of [42]) or when the pressure function is expanded as a cosine series (see equation (12) of [6]).† The requirement that the integral equation be satisfied is equivalent to the boundary residual criterion. Miller [15], for example, took four terms of the cosine expansion and showed that by satisfying the integral equation approximately, he effectively matched the slope of the indenter at four points.

By expanding the kernel of the integral equation in a series form, one can effectively assume the form of the set of base functions. The question naturally arises whether it would be simpler and more accurate to determine the coefficients to the base functions through a least square, collocation or some other boundary residue criterion, rather than through the integral equation. If this is indeed possible, the probable benefits would be: (a) better accuracy, (b) extended applicability to smaller thickness to contact radius ratio and (c) avoidance of further analytical complexity when more than one layer is bonded to the half-space, and also when two multilayered bodies are on contact.

This paper is divided into two basic parts, Sections 2 and 3, respectively.

In the next section (Section 2), the basic formulation for the approximation method is given. Taking five terms of Dhaliwal's solution for a flat-ended punch on one layer bonded to a half-space as the assumed set of base functions, it is shown that the boundary conditions are satisfied to a very high degree of accuracy, and that the accuracy could be improved with an additional term in the assumed base functions. The same numerical experiment was applied to two layers bonded to a half-space and it is demonstrated that the same degree

† This is a simplified description of a wealth of methods and techniques. We cite two examples here for the purpose of illustration.

of high accuracy can be maintained. Subsequently, the technically more useful case of a parabolic shaped punch is considered. For various combinations of material properties, layer thicknesses and number of layers, the stress distributions, contact force and penetration are plotted. The problem of an elastic indenter is also discussed.

Section 3 contains an extension of the analysis to quasi-static impacts, which parallels the original treatment of Hertz. This also furnishes an expedient vehicle for experimental verifications. An experimental study of a steel ball dropped on several different multilayered media is described. The contact times obtained from the experiments were compared with theoretical results obtained through integration of the equation of motion.

2. CONTACT STRESS ANALYSIS

(a) *Boundary conditions*

An elastic half-space is composed of one or two elastic layers bonded to an elastic homogeneous half-space as shown in Fig. 1. The dissimilar materials adhere to each other perfectly. A rigid axisymmetric punch is pressed into the surface of the layered half-space. Assuming frictionless contact, one wishes to find the surface pressure, amount of penetration

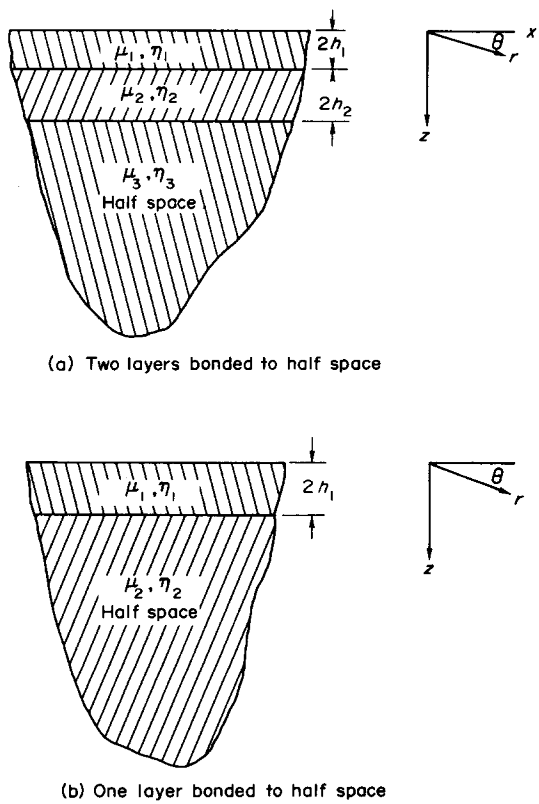

FIG. 1. Sketch of multilayer media.

and contact radius, from a knowledge of the load and the profile of the rigid punch. The surface boundary conditions can be written as

$$u_z(r,0) \equiv a\omega(r) = a[\delta - f(r)], \quad 0 \le r \le a, \tag{1}$$

$$\sigma_{zz}(r,0) = 0, \qquad r > a, \tag{2}$$

$$\sigma_{rz}(r,0) = 0, \qquad r > 0, \tag{3}$$

where a is the radius of contact region, $f(r)$ is the profile of the punch divided by a and δ is the ratio of the penetration of the punch to the contact radius.

Over the interfaces $z = 2h_1$ and $z = 2(h_1+h_2)$ the surface traction and displacements must be continuous, i.e. on $z = 2h_1$

$$\begin{aligned} u_z^{(1)} &= u_z^{(2)}, \\ u_r^{(1)} &= u_r^{(2)}, \\ \sigma_{rz}^{(1)} &= \sigma_{rz}^{(2)}, \\ \sigma_{zz}^{(1)} &= \sigma_{zz}^{(2)}, \end{aligned} \tag{4}$$

and on $z = 2\,(h_1 + h_2)$

$$\begin{aligned} u_z^{(2)} &= u_z^{(3)}, \\ u_r^{(2)} &= u_r^{(3)}, \\ \sigma_{rz}^{(2)} &= \sigma_{rz}^{(3)}, \\ \sigma_{zz}^{(2)} &= \sigma_{zz}^{(3)}. \end{aligned} \tag{5}$$

Finally, at distances far away from the contact region, the stresses and displacements must vanish.

(b) *Approximate method of analysis*

If the elastic half-space were homogeneous, the contact problem specified in the last section could be reduced to a mixed boundary value problem in potential theory. Many such elasticity problems can be solved in closed form. When the elastic half-space is made up of layers of dissimilar materials, the mathematics become more complex and one invariably has to appeal to various methods of approximate analysis. In the present treatment, we shall satisfy the equilibrium equations, and the boundary conditions (2)–(5) exactly, and the surface displacement condition, equation (1) approximately.

Intuitively, if the punch profile remains the same, the contact pressure distribution for the top surface of the layered half-space and that for the homogeneous half-space are similar in many respects. For example, if the punch profile is smooth, then the pressure is always zero at the edge. If the punch has a sharp corner, the pressure will be singular there. For this reason, it appears reasonable to express the contact pressure $q(r)$ by

$$q(r) = \mu_1\delta[p_0 q_0(r) + q_s(r)], \tag{6}$$

where $q_0(r)$ is the contact pressure in the corresponding homogeneous half-space problem, $q_s(r)$ is a "perturbing term" and p_0 a constant as yet undetermined. On the right-hand side of equation (6), the functions $q_0(r)$ and $q_s(r)$ are nondimensional since μ_1, the shear modulus has the dimension of pressure. If the material properties in the other layers are not much different than that of the top layer, δp_0 and $q_s(r)$ will approach unity and zero, respectively. One might suspect that in the layered medium $q_s(r)$ would be a smooth function which vanishes at the edge ($r = a$) and is stationary at the center. If $a\omega_0(r)$ and $a\omega_s(r)$ are the surface displacement functions associated with the pressure distributions $\mu_1 . q_0(r)$, $\mu_1 . q_s(r)$, respectively, the displacement boundary condition, equation (1) becomes

$$1 - f(r)/\delta - p_0\omega_0(r) - \omega_s(r) = 0. \tag{7}$$

The "perturbing" function $q_s(r)$ is now approximated by

$$q_s(r) = \sum_{i=1}^{n} p_i q_i(r), \tag{8}$$

where $q_i(r), i = 1 \ldots n$, form a set of base functions. They have the properties of being smooth continuous functions which vanish at the edge and are stationary at the center. The error from this approximation then becomes

$$\varepsilon(r) = \left[1 - f(r)/\delta - \sum_{i=0}^{n} p_i\omega_i(r)\right], \tag{9}$$

where $a\omega_i(r)$ are the surface displacements corresponding to the surface pressure $\mu_1 q_i(r)$. Some of the logical schemes to determining δ and p_i, are the collocation or point matching method, the integral least square method and the Reissner energy principle.

In the standard collocation or point matching approach δ and p_i are chosen such that the error is zero.

$$\varepsilon(r_i) = 0, \tag{10}$$

over $(n+2)$ number of points.

The integral least square approach requires that the coefficients δ and p_i are such that the integral

$$I_L = \frac{1}{a^2}\int_0^a r\varepsilon^2(r)\,\mathrm{d}r, \tag{11}$$

is a minimum, i.e.

$$\frac{\partial I_L}{\partial \delta} = 0, \qquad \frac{\partial I_L}{\partial p_i} = 0, \qquad (i \leq n+1). \tag{12}$$

A hybrid method can also be used, where more than $(n+2)$ collocation points are chosen. Here one seeks a least square solution of an overdetermined system of linear equations [46].

In applying the Reissner energy principle, one minimizes the integral

$$I_R = \frac{1}{a^2}\int_0^a \varepsilon(r)\sum_0^n p_i q_i(r) r\,\mathrm{d}r. \tag{13}$$

In all of the three methods $(n+2)$ linear algebraic equations result for determining the unknown coefficients, δ and p_i.

From a physical point of view, the Reissner energy method is most appealing. However, there is some computational difficulty in cases where the punch contains a sharp corner. Numerical experiments (which will be discussed later) have indicated that the integral least square method is more accurate than the collocation method for the same computational effort. For this reason, the integral least square method is employed in this paper.

A very important factor influencing the accuracy of the approximation scheme is the choice of the set of approximating base functions $q_i(r)$. It is chosen to be

$$q_i(r) = (1-r^2/a^2)^{i-\frac{1}{2}}, \; i = 1, 2 \ldots n. \tag{14}$$

It will be demonstrated that, by approximating the pressure distribution as

$$q(r) = \delta\mu_1 \sum_{i=0}^{n} p_i q_i(r) \tag{15}$$

where $q_0(r)$ is the pressure for the homogeneous half-space problems and $q_i(r)$ ($i = 1 \ldots n$) are of the forms in equation (14), very high accuracy could be achieved.

A primary prerequisite for the numerical method is to compute the surface displacement $\omega_i(r)$ induced by the pressure $q_i(r)$, on the layered half-space. An efficient and accurate method for this computation has been presented in [47].† Double precision arithmetic was employed in the computation of $\omega_i(r)$ and special precautions were taken to ensure that the numerical values of $\omega_i(r)$ used in this paper are accurate to 14–15 significant figures.

(c) *Indentation by a circular flat ended punch*

When the end of the punch is flat, $f(r) = 0$, and the boundary condition is given by

$$\omega(r) = \delta \qquad 0 \leq r < a. \tag{16}$$

The solution for the homogeneous half-space problem is well known. From equations (14) and (15), the surface pressure is approximated by

$$q(r) = \delta\mu_1 \sum_{i=0}^{n} p_i (1-r^2/a^2)^{i-\frac{1}{2}}. \tag{17}$$

Aside from engineering interest, the purpose of doing this numerical example is twofold: (a) Since the punch has a sharp corner with a stress singularity there, it would serve as a touchstone to the approximate method. (b) Dhaliwal [42] has studied this problem for the special case of a single layer bonded to a homogeneous half-space. Dhaliwal's paper contains some numerical results and thus affords an excellent opportunity for comparison with the present investigation.

Since $f(r) = 0$, equations (9), (11) and (12) lead to a system of $(n+1)$ linear algebraic equations:

$$\frac{1}{a^2} \int_0^a r \left[1 - \sum_{i=0}^{n} p_i \omega_i(r)\right] \omega_j(r) \, dr = 0, \qquad j = 0, 1 \ldots n. \tag{18}$$

The total load is obtained from integrating equation (17),

$$P = 2\pi\mu_1 \delta a^2 \sum_{i=0}^{n} p_i \Big/ (1+2i). \tag{19}$$

The integration in equation (18) is carried out through Gaussian quadrature methods, taking numerical values of $\omega_i(r)$ computed from the method discussed in [47].

† The formulae used for the computations of $\omega_i(r)$ are given in the Appendix.

In Dhaliwal's analysis, the representation of contact pressure, in the case of a thick layer, can be written in the same form as in equation (17), with $n = 4$. Thus, one can compare the total loads as well as the pressure distributions obtained from the two methods. Also, one can compute the actual surface displacements corresponding to the calculated pressure distributions to see how accurately the boundary condition equation (16) is approximated. Numerical results for three different thicknesses and five different values of shear modulus are shown in Table 1. For each thickness and material property combination, two sets of values of P and p_i ($i = 0, 1, 2, 3, 4$) are listed: the first line is calculated from the integral least square method, the 2nd line is calculated from Dhaliwal's analysis.†

Figures 2 and 3 give the errors in the surface displacement variation when the surface pressure is represented in the form of equation (17), for the cases of $2h_1/a = 3, 4$, $\mu_1/\mu_2 = 5$, $\nu_1 = 0{\cdot}333$, $\nu_2 = 0{\cdot}250$. The maximum error from the present method is less than 10^{-10}. The maximum error obtained from Dhaliwal's approximate solution to the exact integral equation formulation is of the order of 0·01 for $2h_1/a = 4$, and amounts to 0·17 for $2h_1/a = 3$.

In Figs. 2 and 3, the layers are relatively thick. Dhaliwal's representations break down for thinner layers. However, this is not a limitation in our approximate method. A better appreciation of the capability and limitation of the present method may be obtained through numerical computation of thinner layers. In Figs. 4 and 5, there are two thin layers ($2h_1/a = 2h_2/a = 0{\cdot}2$) bonded to the elastic half-space. The maximum errors from the present method are of the order of 10^{-4}. In these two graphs, the corresponding error curves from a five point collocation scheme equation (10) are also presented. Although the errors from the collocation scheme are perfectly acceptable, the integral least square method is clearly superior.

Figure 6 shows the pressure distribution under the flat ended punch when there is one hard layer bonded to a soft substrate ($\mu_2/\mu_1 = 0{\cdot}1$), for various thickness to radius of contact ratio. When $2h_1/a = 2$, the surface pressure does not differ much from the Boussinesq solution. When the layers are thinner ($2h_1/a = 0{\cdot}5, 1$), there is a central region where the normal stress between the layer and the punch is tensile. When the layer is still thinner, ($2h_1/a = 0{\cdot}3$), this region of tensile stress becomes an annular strip. It is implicitly assumed in contact stress analysis that the normal pressure between the layer and the punch is always compressive, i.e. there is no separation. The examples in Fig. 6 demonstrate that in the case of a thin layer bonded to a soft substrate, there is a circular or annular region of separation.

(d) *Parabolic punch*

The capability and limitation of the proposed approximating base functions coupled with the integral least square method were demonstrated previously. Of more practical interest is the stress analysis of a parabolic punch pressed into a layered half-space. The

† We were led to repeat Dhaliwal's numerical calculations due to substantial disagreement between our results and his published results. We found that all his numbers in Table 4 and Figs. 1–3 of his paper should be divided by the factor $4/(1-\nu_1)$. With that correction made, we recomputed row 6 to row 14 of his Table 4 from his Table 1 with double precision arithmetic. It was found that from row 7 on, except for one obvious error at $\mu = 1$, $2h = 25$, we could substantially reproduce his numerical results. We could not, however, reproduce the numbers in row 6 (i.e. $2h = 3$). It is believed that the numerical values in row 6 of Table 4 in Dhaliwal's paper are in error in the second and sometimes in the first significant figure, e.g. compare Table 1 here to Table 4 of Dhaliwal's paper. Rows 1–5 of Table 4 in Dhaliwal's paper are computed from a different method which does not give pressure distributions. Our results agree with his in the general trend, but differ in numerical values up to 10–15 per cent.

TABLE 1. NORMALIZED FORCE P AND THE PRESSURE COEFFICIENT p_i OF A FLAT ENDED PUNCH ON ONE LAYER BONDED TO A HALF-SPACE. *CE* REPRESENTS RESULTS FROM METHOD IN THIS PAPER, *IEA* REPRESENTS RESULTS COMPUTED BY CHEN AND ENGEL BASED INTEGRAL EQUATION APPROXIMATE SOLUTION (DHALIWAL [42])

Method	μ	P	p_1	p_2	p_3	p_4	p_5
$h = 3$							
CE	0·25	7·49393	1·17877	3·9994E-02	2·8666E-03	1·4012E-04	5·9963E-06
IEA	0·25	7·49318	1·17851	4·0428E-02	2·8422E-03	1·3326E-04	9·4980E-06
CE	0·50	6·85144	1·08267	2·2330E-02	1·5667E-03	7·5750E-05	3·2245E-06
IEA	0·50	6·85115	1·08258	2·2478E-02	1·5702E-03	6·6562E-05	5·5917E-06
CE	3·00	4·09096	0·66294	−3·4307E-02	−1·9911E-03	−8·8520E-05	−3·6229E-06
IEA	3·00	4·08996	0·66261	−3·3751E-02	−2·0948E-03	−4·3342E-05	−1·0529E-05
CE	4·00	3·61186	0·58861	−3·9913E-02	−2·2238E-03	−9·7260E-05	−3·9531E-06
IEA	4·00	3·57013	0·58222	−4·0539E-02	−2·4876E-03	−6·3974E-05	−1·3023E-05
CE	5·00	3·25696	0·53318	−4·3012E-02	−2·3158E-03	−9·9960E-05	−4·0414E-06
IEA	5·00	2·70273	0·44882	−5·3988E-02	−3·2863E-03	−1·0588E-04	−1·4775E-05
$h = 4$							
CE	0·25	7·08683	1·12189	1·7571E-02	7·5110E-04	2·2358E-05	5·7594E-07
IEA	0·25	7·08673	1·12186	1·7612E-02	7·4890E-04	2·2371E-05	7·1314E-07
CE	0·50	6·63352	1·05233	1·0026E-02	4·1974E-04	1·2370E-05	3·1735E-07
IEA	0·50	6·63348	1·05232	1·0040E-02	4·1946E-04	1·2070E-05	4·1985E-07
CE	3·00	4·42373	0·70983	−1·6940E-02	−5·8964E-04	−1·6071E-05	−3·9918E-07
IEA	3·00	4·42366	0·70980	−1·6883E-02	−5·9565E-04	−1·3498E-05	−7·9060E-07
CE	4·00	3·98883	0·64167	−2·0068E-02	−6·7131E-04	−1·8022E-05	−4·4522E-07
IEA	4·00	3·98624	0·64126	−2·0071E-02	−6·8743E-04	−1·6146E-05	−9·7780E-07
CE	5·00	3·65508	0·58918	−2·1924E-02	−7·0934E-04	−1·8814E-05	−4·6286E-07
IEA	5·00	3·61993	0·58381	−2·2573E-02	−7·7060E-04	−1·9404E-05	−1·1093E-06
$h = 5$							
CE	0·25	6·51117	1·03554	2·2278E-03	2·5249E-05	2·0286E-07	1·3715E-09
IEA	0·25	6·51117	1·03554	2·2279E-03	2·5246E-05	2·0354E-07	1·3928E-09
CE	0·50	6·30760	1·00345	1·3129E-03	1·4582E-05	1·1609E-07	7·7850E-10
IEA	0·50	6·30760	1·00345	1·3130E-03	1·4581E-05	1·1618E-07	8·2000E-10
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IEA	3·00	5·07790	0·80906	−2·6356E-03	−2·4467E-05	−1·7858E-07	−1·5441E-09
CE	4·00	4·77507	0·76106	−3·2466E-03	−2·8986E-05	−2·1148E-07	−1·3733E-09
IEA	4·00	4·77511	0·76107	−3·2463E-03	−2·9004E-05	−2·0982E-07	−1·9098E-09
CE	5·00	4·52490	0·72139	−3·6587E-03	−3·1624E-05	−2·2816E-07	−1·4752E-09
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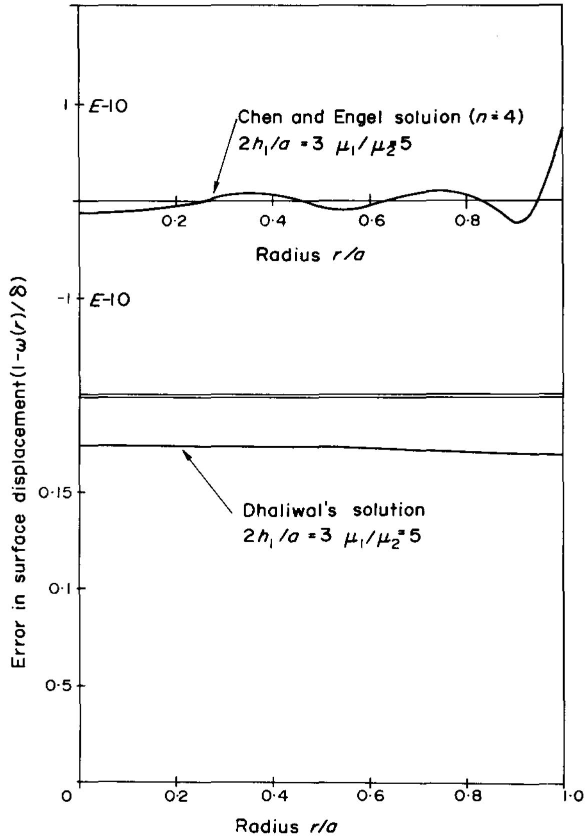

FIG. 2. Error comparison, flat ended punch, $2h_1/a = 3$, $n = 4$.

boundary condition is

$$u(r, z) = a\omega(r) = a\delta - \frac{r^2}{2R}. \tag{20}$$

The parabolic shape is used to approximate a spherical body of radius R, where $R \gg a$. From equations (14) and (15), the surface pressure is approximated by

$$q(r) = \delta\mu_1 \sum_{i=1}^{n} p_i(1-r^2/a^2)^{i-\frac{1}{2}}. \tag{21}$$

From equations (9) and (20), the error from the approximation is

$$\varepsilon(r) = 1 - \beta r^2/a^2 - \sum_{i=1}^{n} p_i\omega_i(r), \tag{22}$$

where β is defined as

$$\beta = a/2R\delta. \tag{23}$$

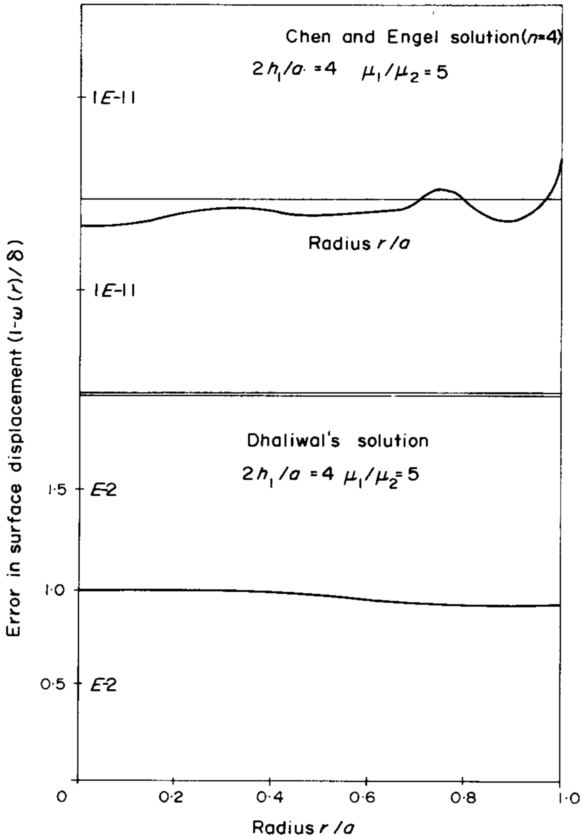

FIG. 3. Error comparison, flat ended punch, $2h_1/a = 4$, $n = 4$.

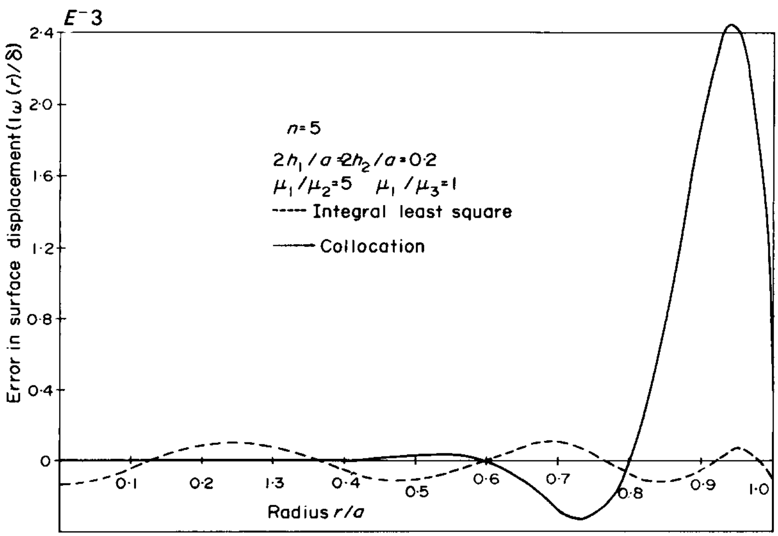

FIG. 4. Errors for thin layers, $2h_1/a = 2h_2/a = 0{\cdot}2$.

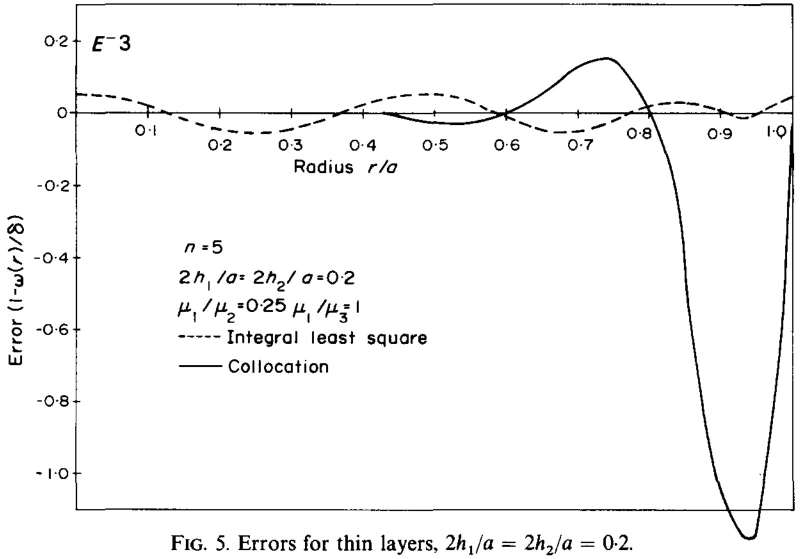

FIG. 5. Errors for thin layers, $2h_1/a = 2h_2/a = 0{\cdot}2$.

The n coefficient p_i and the parameter β are computed from the following $(n+1)$ linear algebraic equations

$$3-2\beta-\frac{12}{a^4}\int_0^a r^3 \sum_{i=1}^{n} p_i\omega_i(r)\,\mathrm{d}r = 0, \tag{24}$$

$$\int_0^a r\omega_j(r)\left[1-\beta r^2/a^2-\sum_{i=1}^{n} p_i\omega_i(r)\right]\mathrm{d}r = 0, (j = 1, 2, \ldots n). \tag{25}$$

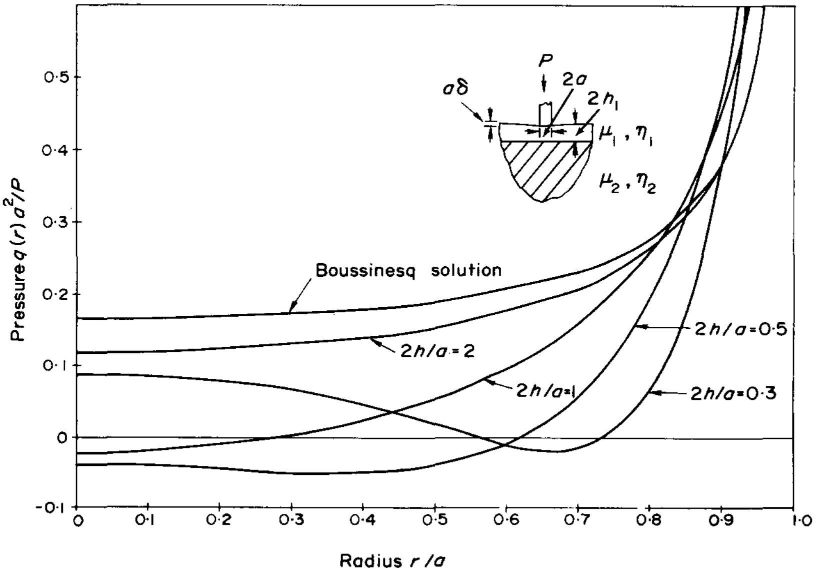

FIG. 6. Dimensionless pressure distribution under the flat ended punch, $q(r)a^2/P$. $\nu_1 = \nu_2 = \frac{1}{3}$, $\mu_2/\mu_1 = 0{\cdot}1$.

The total load is

$$P = 2\pi\mu_1\delta a^2 \sum_{i=1}^{n} p_i \Big/ (1+2i). \tag{26}$$

In all of the numerical computations on the parabolic punch, n was taken to be 5.

For comparison purposes, it is convenient to plot the nondimensional quantities, defined as follows:

$$P^* \equiv \frac{3(1-\nu_1)PR}{8\mu_1 a^3} = \frac{3\pi(1-\nu_1)}{8\beta}\sum_{i=1}^{n}\frac{p_i}{1+2i}, \tag{27}$$

$$\delta^* \equiv \frac{R\delta}{a} = \frac{1}{2\beta}. \tag{28}$$

The physical meaning of P^* is that $(P^*)^{1/3}$ is equal to a_H/a where a_H is the Hertz contact radius for a medium of shear modulus μ_1 and Poisson's ratio ν_1.

In the case of a homogeneous half-space, both P^* and δ^* are equal to 1. It follows that when the layers are very thick or very thin, δ^* would approach 1 in both cases; and P^* would approach 1, or μ_j/μ_1, where μ_j is the shear modulus of the base.

In Table 2 are shown the numerical values of P^* and δ^* computed from a single layer bonded to a soft ($\mu_1/\mu_2 = 3$ and 10) stratum or to a hard ($\mu_2/\mu_1 = 3$ and 10) stratum. They

TABLE 2. THE NONDIMENSIONALIZED LOAD P^* AND PENETRATION δ^* FOR ONE LAYER BONDED TO A HALF-SPACE. $\nu_1 = \nu_2 = \frac{1}{3}$

μ_2/μ_1	10		3		1/3		1/10	
$2h_1/a$	P^*	δ^*	P^*	δ^*	P^*	δ^*	P^*	δ^*
∞	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
16·000	1·0001	0·9669	1·0001	0·9763	0·9999	1·0571	0·9997	1·2000
8·000	1·0009	0·9345	1·0006	0·9531	0·9989	1·1132	0·9975	1·3968
4·000	1·0070	0·8746	1·0048	0·9101	0·9918	1·2174	0·9808	1·7632
2·000	1·0469	0·7815	1·0318	0·8439	0·9477	1·3684	0·8811	2·2642
1·500	1·0954	0·7389	1·0640	0·8149	0·8998	1·4189	0·7812	2·3884
1·000	1·2310	0·6868	1·1509	0·7834	0·7936	1·4342	0·5891	2·3224
0·600	1·5675	0·6479	1·3456	0·7117	0·6381	1·3568	0·3695	1·9572
0·400	1·9975	0·6406	1·5557	0·7845	0·5415	1·2720	0·2631	1·6729
0·300	2·3968	0·6466	1·7194	0·8016	0·4923	1·2191	0·2164	1·5176
0·200	3·0804	0·6672	1·9474	0·8304	0·4442	1·1610	0·1754	1·3600
0·150	3·6341	0·6880	2·0971	0·8511	0·4203	1·1297	0·1567	1·2794
0·100	4·4894	0·7246	2·2897	0·8800	0·3937	1·0920	0·1373	1·1876
0·000	10·0000	1·0000	3·0000	1·0000	0·3333	1·0000	0·1000	1·0000

are plotted in Figs. 7 and 8. It appears from Fig. 7 that for $2h_1/a \geq 4$ the surface force on the thick layer behaves essentially like a homogeneous half-space, but the penetration does not. This shows that in design problems relating to layered media, it makes a substantial difference whether interference is prescribed or whether force is prescribed. The difference in surface forces between a homogeneous and layered medium becomes really significant when the thickness ratio $2h_1/a$ is less than 2.

Table 3 shows the numerical values of P^* and δ^* computed from two layers bonded to a half-space with a soft ($\mu_1/\mu_2 = 3$ and 10) or hard ($\mu_2/\mu_1 = 3$ and 10) interlayer. They are

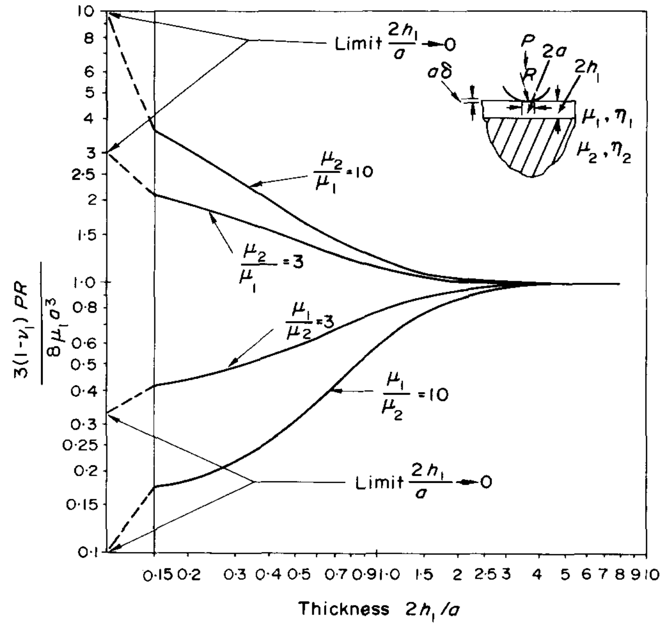

FIG. 7. Variation of $[3(1-\nu_1)PR]/8\mu a$ with $2h_1/a$ for a single layer bonded to a half-space. $\nu_1 = \nu_2 = \frac{1}{3}$.

plotted in Figs. 9 and 10. Again it is apparent that the surface force on the thick layer is essentially like that of a homogeneous half-space for $2h_1/a \geq 4$. The effect of the interlayer becomes strongest at $2h_1/a \approx 0{\cdot}3$–$0{\cdot}5$, and very slowly approaches the homogeneous half-space situation again as the thicknesses decrease. Figure 10 shows that when the interlayer

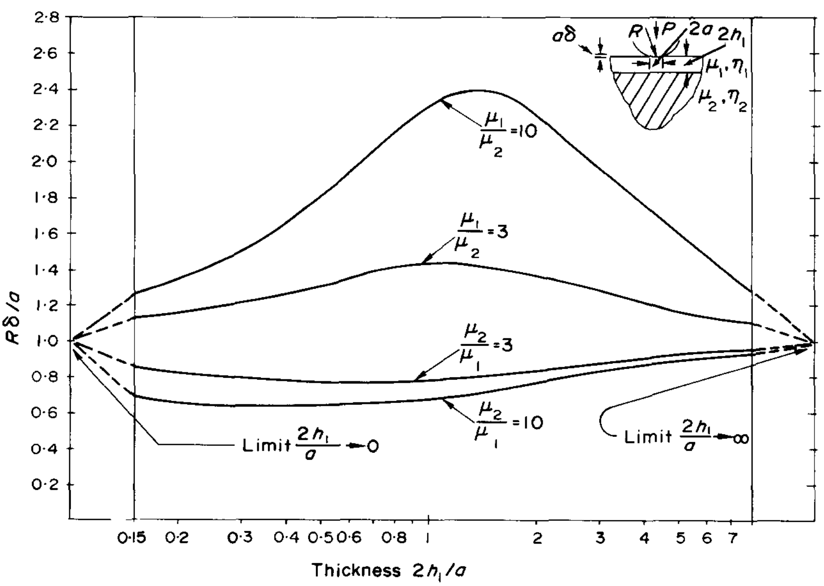

FIG. 8. Variation of $R\delta/a$ with $2h_1/a$ for a single layer bonded to a half-space. $\nu_1 = \nu_2 = \frac{1}{3}$.

TABLE 3. THE NONDIMENSIONALIZED LOAD P^* AND PENETRATION δ^* FOR TWO LAYERS BONDED TO A HALF-SPACE. $h_1 = h_2, \mu_1 = \mu_3, \nu_1 = \nu_2 = \nu_3 = \frac{1}{3}$

μ_2/μ_1 / $2h_1/a$	10		3		1/3		1/10	
	P^*	δ^*	P^*	δ^*	P^*	δ^*	P^*	δ^*
∞	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
16·000	1·0001	0·9802	1·0001	0·9871	0·9999	1·0256	0·9997	1·0766
8·000	1·0009	0·9612	1·0006	0·9747	0·9991	1·0503	0·9979	1·1510
4·000	1·0065	0·9278	1·0043	0·9532	0·9931	1·0937	0·9842	1·2835
2·000	1·0426	0·8887	1·0277	0·9290	0·9566	1·1415	0·9027	1·4325
1·500	1·0847	0·8839	1·0543	0·9275	0·9181	1·1431	0·8212	1·4396
1·000	1·1921	0·9115	1·1184	0·9480	0·8375	1·1048	0·6645	1·3379
0·600	1·3775	1·0182	1·2151	1·0125	0·7402	1·0108	0·4937	1·1077
0·400	1·4483	1·1095	1·2430	1·0596	0·7100	0·9487	0·4323	0·9581
0·300	1·4132	1·1358	1·2239	1·0727	0·7152	0·9232	0·4237	0·8878
0·200	1·3037	1·1207	1·1717	1·0673	0·7471	0·9109	0·4466	0·8325
0·150	1·2327	1·0981	1·1358	1·0568	0·7779	0·9132	0·4794	0·8159
0·100	1·1639	1·0717	1·1002	1·0447	0·8173	0·9201	0·5332	0·8078
0·000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000

is hard ($\mu_2/\mu_1 = 3$ and 10) and $2h_1/a$ is large, the value of the surface penetration parameter δ^* is < 1, similar to the soft layer over the hard stratum situation in Fig. 8; and when $2h_1/a$ is small, $\delta^* > 1$, since the two top layers may be thought of as an equivalent hard layer on a soft stratum. The reverse effects occur when the interlayer is soft.

Figure 11 is a plot of the normalized surface pressure under the parabolic punch for $2h_1/a = 0{\cdot}3$. The lesson to be learned here is that when there is a soft interlayer (or stratum), there is substantial deviation from the Hertz solution. Particularly interesting are the cases

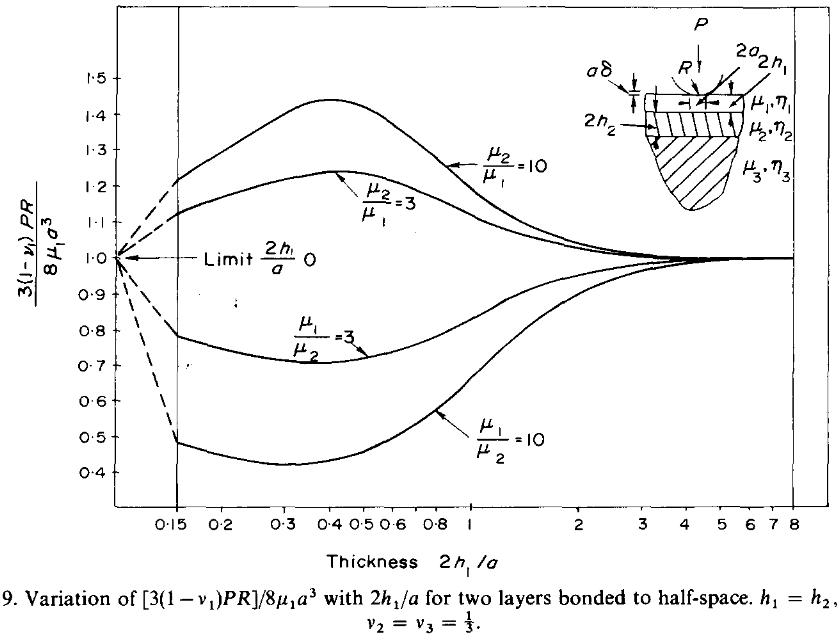

FIG. 9. Variation of $[3(1-\nu_1)PR]/8\mu_1a^3$ with $2h_1/a$ for two layers bonded to half-space. $h_1 = h_2$, $\nu_1 = \nu_2 = \nu_3 = \frac{1}{3}$.

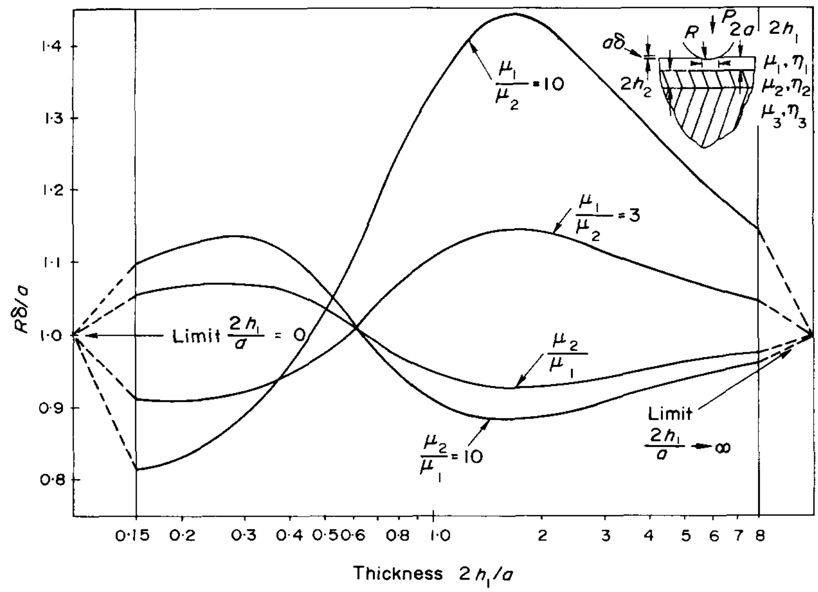

FIG. 10. Variation of $R\delta/a$ with $2h_1/a$ for two layers bonded to a half-space. $h_1 = h_2$. $\nu_1 = \nu_2 = \nu_3 = \frac{1}{3}$.

when the substrate is soft. It is demonstrated that the maximum pressure need not be at the center.

In practice, both of the contacting bodies are elastic and may have finite radii of curvature. In this case, it should also be possible to establish an asymptotic thick layer solution to the integral equation formulation of the elasticity problem. That this is not a trivial

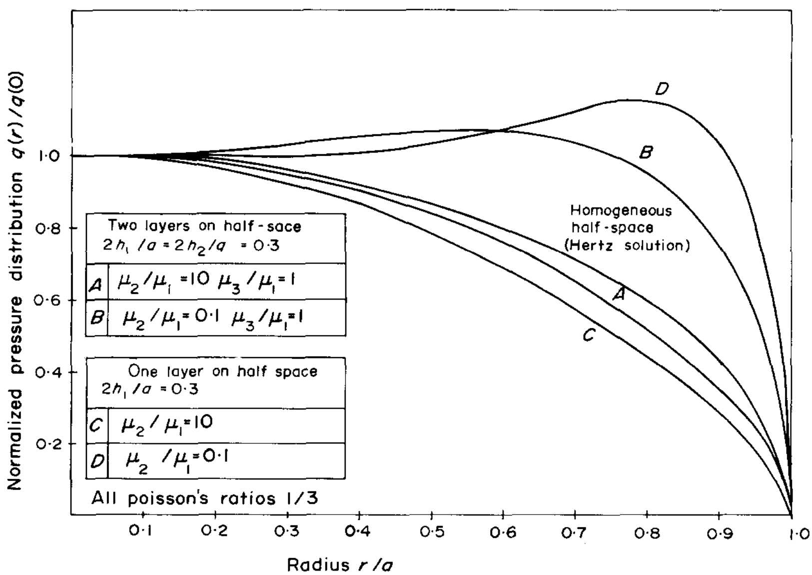

FIG. 11. Normalized surface pressure distribution under parabolic punch $q(r)/q(0)$.

matter is clear from the work of Keer [13]. However, in the present approximate scheme, one merely replaces $\omega_i(r)$ and β in equations (22) and (23) by

$$\omega_i(r) = \omega_i^{\mathrm{I}}(r) + \omega_i^{\mathrm{II}}(r) \tag{29}$$

$$\beta = \frac{a}{2\delta}\left(\frac{1}{R_{\mathrm{I}}} + \frac{1}{R_{\mathrm{II}}}\right) \tag{30}$$

where $\omega_i^{\mathrm{I}}(r)$ and ω_i^{II} are the surface displacements in the two contacting bodies induced by the pressure $q_i(r)$ and R_{I} and R_{II} are the two radii of curvature. It is an easy matter to implement these two changes in the computer programs.

3. IMPACT ON A MULTILAYER MEDIUM

(a) *Method of analysis*

The results obtained in the last section can be applied to the problem of impact of two solid bodies, along the quasi-static assumption of the Hertz impact theory. Accordingly, it will be assumed that the surface compresses gradually. The dissipation of energy during impact is neglected. We shall determine the variation of contact force and surface displacement as a function of time, when a rigid ball of radius R and weight W is dropped from a height H on the layered medium.

As the ball indents the medium, the contact radius is continuously changing with time. One may write the contact force P as a function of the dimensionless ratio a/h_1,

$$P = \frac{\mu_1 h_1^3}{2R} F\left(\frac{a}{h_1}\right), \tag{31}$$

and from equation (27), this function $F(a/h_1)$ is

$$F\left(\frac{a}{h_1}\right) = \frac{16a^3 P^*}{3(1-\nu_1)h_1^3} = \frac{2\pi a^3}{\beta h_1^3} \sum_{i=1}^{n} \frac{p_i}{1+i}. \tag{32}$$

Similarly, the central surface displacement $u \equiv u_z(0, 0)$ may be considered as a function of the ratio a/h_1,

$$u = \frac{h_1^2}{2R} G\left(\frac{a}{h_1}\right), \tag{33}$$

and from equation (28) this function $G(a/h_1)$ is

$$G\left(\frac{a}{h_1}\right) = \frac{2a^2\delta^*}{h_1^2} = \frac{a^2}{h_1^2\beta}. \tag{34}$$

When the ball and the elastic half-space are in contact, the equation of motion of the ball is

$$\frac{W}{g}\frac{\mathrm{d}v}{\mathrm{d}t} = W - P \tag{35}$$

where v is the surface velocity at $r = 0$, $z = 0$.

Combining (31), (33) and (35) the following system of first order simultaneous differential equations is obtained,

$$\frac{\mathrm{d}(a/h_1)}{\mathrm{d}t} = \frac{2R}{h_1^2}\frac{v}{G'(a/h_1)}, \tag{36a}$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = v, \tag{36b}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g\left[1 - \frac{\mu_1 h_1^3}{2RW}F\left(\frac{a}{h_1}\right)\right], \tag{36c}$$

subjected to the initial conditions,

$$t = 0, \qquad a/h_1 = 0, \qquad u = 0, \qquad v = \sqrt{2gH}. \tag{37}$$

This set of first order equations could be integrated numerically in a straightforward manner, were it not for the fact that at the first instant of impact $G'(a/h_1) = 0$.

To circumvent this difficulty, Hertz impact on a homogeneous half-space is considered for two very small time elements Δt_0 utilizing the differential equations

$$\frac{\mathrm{d}u}{\mathrm{d}t} = v \tag{38a}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -g\left(\frac{8\mu_1 R^{\frac{1}{2}} u^{\frac{3}{2}}}{3(1-\nu_1)W} - 1\right), \tag{38b}$$

and the initial condition, $t = 0$; $u = 0$; $v = \sqrt{2gH}$. After $2\Delta t_0$, the equations (36) are employed with the initial conditions,

$$\text{at} \qquad t = 2\Delta t_0 : a/h_1 = \sqrt{\frac{2Ru(2\Delta t_0)}{h_1^2}}, \qquad u = u(2\Delta t_0), \qquad v = v(2\Delta t_0). \tag{39}$$

The functions $F(a/h_1)$ and $G(a/h_1)$ are computed through equations (32) and (34) according to the approximate method described in the last section. Numerical integration of (38) and (36) is carried out through a 4th order Runge–Kutta scheme. The contact time stretches from the initial condition until the surface displacement u returns to zero.

(b) *Experimental work*

The objective of the experimental work was the verification of contact durations obtained analytically for a given configuration of ball and elastic layered medium. An important constraint on the choice of materials is the consideration that all materials involved should stay in the elastic range during impact.

For an experimental study involving layers of infinite lateral extent, the ball dropping method appears straightforward. Since the bonds between the strata eliminate the low frequency effects, the quasi-static condition postulated in the mathematical analysis is maintained. The elastic bulk properties of the individual layers can be obtained from Hopkinson bar experiments on longitudinal specimens. The phase velocity V is measured, and the Young's modulus is then computed from $V = \sqrt{E/\rho}$. Even for strongly viscoelastic materials of large time constants, the modulus of elasticity thus obtained is suitable for approximate elastic analysis.

It has been found that, as a top layer, nylon is suitable for the purpose of rendering a linearly elastic medium. The highly elastic properties of nylon are due to its hydrogen bonded molecular structure.

The mathematical analysis prescribes a condition of complete adherence between layers. In the experimental setup, this condition can only be approximated, by the device of either thinly applied adhesives or thin double-sided adhesive tape between the layers. In addition to this departure from the ideal situation, the experimental measurement of contact time necessitates a microscopic layer of silver spray to be deposited on the top surface exposed to the impact. It is within the capability of the present analytical program to check the influence of the presence of all these experimental devices, so as to establish the correct correspondence of test and analysis.

In the experimental setup, as sketched in Fig. 12, a steel ball of 1 in. diameter was dropped on the layered medium and the contact duration measured. This was accomplished by

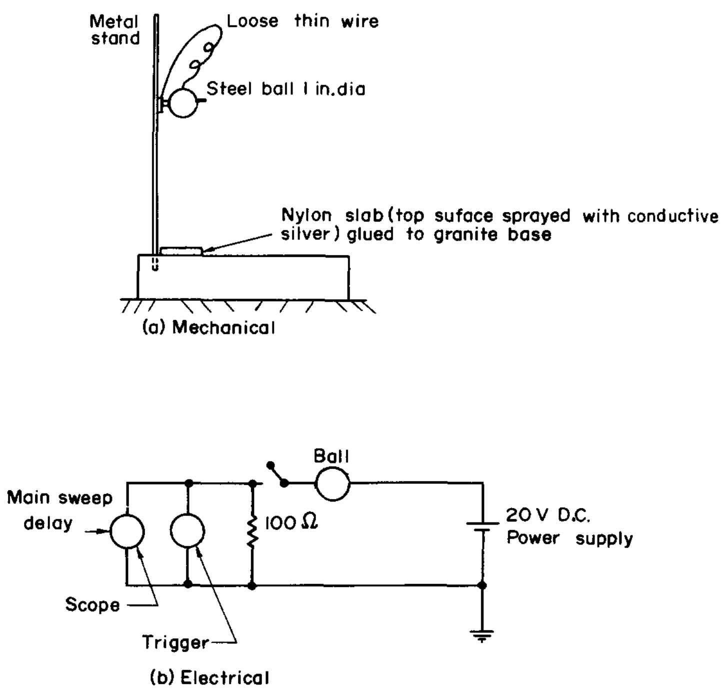

FIG. 12. Mechanical assembly and electrical schematic of test.

connecting the ball in a 20 V d.c. circuit through a thin loose copper wire. The top of the layered medium was sprayed with conductive silver acrylic spray and when the ball hit the surface the circuit closed. The closing of the circuit triggered an oscilloscope and the trace displayed was photographed by a polaroid camera. To eliminate the recording of consecutive rebounds, the main sweep delay feature of the scope was utilized.

As a bottom material, a thick solid granite block was used. Fixed to the support block a steel stand was provided, with a bracket attachable to it at variable heights. This flexible bracket contained a circular hole large enough to let the ball through when the bracket was depressed but able to hold the ball otherwise. To trigger the impact, the ball would be set in the bracket and then the latter slowly depressed to make the ball fall through the hole.

Two major experimental configurations were used, involving two or three layers (including the bottom "half-space"), respectively. In all configurations, the top layer was nylon, in $2\frac{1}{2} \times 2\frac{1}{2}$ in. slabs of variable thickness, machined from $\frac{3}{8}$ in. thick stock. To the granite slab the nylon layers were stuck by Eastman 910 adhesive glue which provides a thin but tough layer approximately 0·0005 in. thick. In the three layer configuration, a soft layer was interposed between the granite pad and the topmost nylon slab; neoprene rubber of durometer 30 was used for this purpose. The neoprene layer (also $2\frac{1}{2} \times 2\frac{1}{2}$ in. in plane, with varied thickness) would be stuck to the foundation by Eastman 910 adhesive glue and to the top nylon slab, by double-sided adhesive tape. This allowed rapid changing of the top layer during an experimental series.

(c) *Experimental and analytical results*

In the ball dropping experiments, the drop height was fixed at 5·12 in. from the top of the composite medium. The top layer (nylon) was varied between $\frac{3}{32}$ and $\frac{8}{32}$ in. There were no permanent deformations in the nylon layers due to the dropping of the 1-in. diameter ball. A faint trace of the contact region was observable on the sprayed surface without, however, any alteration of either the mechanical resistance or continuity of the silver. Computations based on a silver, nylon and granite configuration showed that the presence of the thin conducting coating is indeed negligible for stress analysis. Static shore D durometer tests of the surface showed no appreciable increase in hardness of this coating. Thus, the coating was inconsequential from the point of view of impact results.

First, the two-layer configuration (nylon, granite) is discussed. From previous Hopkinson bar measurements, $V = 66{,}700$ in./sec was obtained for the nylon material, which

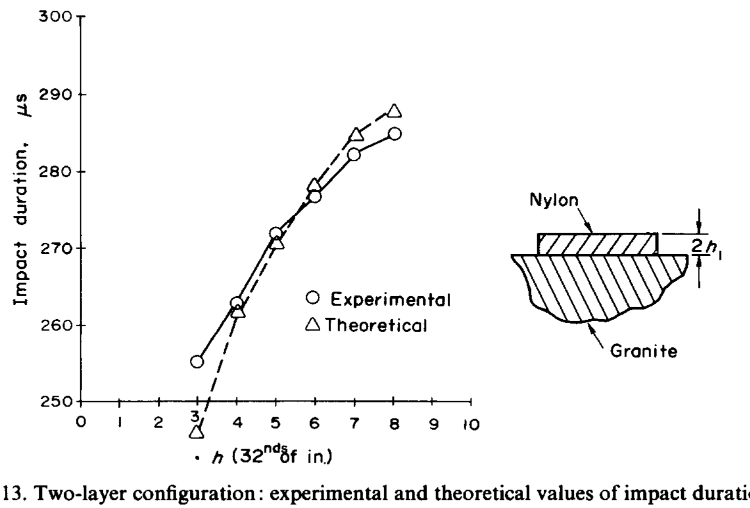

FIG. 13. Two-layer configuration: experimental and theoretical values of impact duration.

gives $E = 475{,}000$ psi. Granite was considered with the elastic properties $E = 7{,}500{,}000$ psi and $\nu = 0{\cdot}1$. Shear moduli are obtained by $\mu = E/[2(1+\nu)]$. Figure 13 shows a graph of contact time vs top layer thickness. Experimental and analytical results are close. The increase in contact time versus increasing top layer thickness demonstrates a decrease in stiffness in the system. The curve flattens out toward a plateau at larger thicknesses, indicating that the condition of a nylon half-space is essentially reached.

Computationally, the two-layer situation is easily handled, since varying the top layer thickness requires the computation of only one set of functions $F(a/h_1)$ and $G(a/h_1)$. The practical range for the nylon-granite configuration was found to be $0 < a/h_1 < 3$, and discrete increments of 0·1 yielded satisfactory continuous description of these functions.

The second case studied was the three layer scheme of a hard layer sitting on a soft one which in turn is supported on a hard half-space. It was realized by using a nylon–neoprene–granite configuration, with two alternative thicknesses of neoprene: $\frac{1}{8}$ and $\frac{1}{2}$ in., respectively. For each of the two middle-layer thicknesses, the top layer thickness was again varied between $\frac{3}{32}$ and $\frac{8}{32}$ in. in $\frac{1}{32}$ in. increments.

A special consideration concerns the Poisson's ratio of rubber. For this nearly incompressible material, ν is obviously close to 0·5. With $E = 130$ psi and $\mu = 43$ psi, ν was first taken equal to 0·495 and then to 0·4995, without producing any substantial change in the outcome of the numerical computations.

The curves of experimental and theoretical results shown in Fig. 14 are in good agreement. It is evident that when the top layer is thin its behavior is largely due to flexure, i.e. thin plate action. However, the action is increasingly described by a localized, contact type phenomenon as the top layer is thickened. These effects are even accentuated as the middle layer is made thicker.

For the computation of the three-layer situation, the numerical work is many times of that for the two-layer configuration. For each top layer thickness, separate sets of $F(a/h_1)$ and $G(a/h_1)$ functions must be generated now even as the middle slab thickness h_2 is kept constant.

4. SUMMARY AND COMMENTS

An approximate method has been presented for the contact stress analysis of a multilayer medium consisting of two (or one) layers bonded to an elastic half-space. The accuracy, and range of applicability of the method is demonstrated through a flat ended punch example. Taking a six term representation, it is shown that for the case of a thick layer ($2h/a > 4$), the maximum in surface displacement error was not much higher than the round-off error in the computer, and when the layer is thin ($2h/a \approx 0{\cdot}2$), the maximum error was of the order of 0·1 per cent. It was also shown that when the substrate is soft, the surface pressure could become tensile at some points, indicating a tendency for separation between the punch and the elastic medium.

Then the parabolic punch was considered. This involved a few simple changes from the previous case (and some slight alterations in the computer program). Numerical results were obtained to illustrate the effects of the different elastic properties and of the thicknesses of the layers. The analysis was then extended to treat quasi-static impact on the multilayer media. Some impact tests on multilayer media were performed and the contact times were measured. These results correlated very well with analytical predictions.

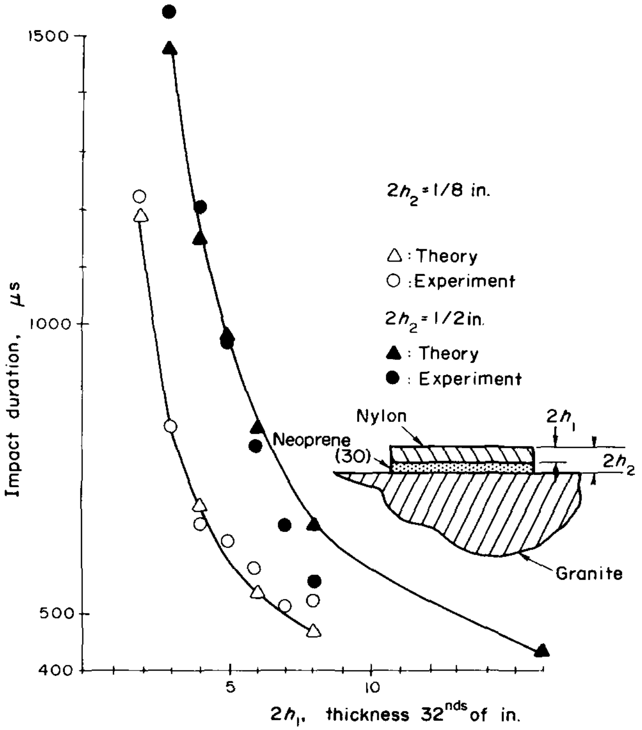

FIG. 14. Three-layer configuration: experimental and theoretical values of impact duration.

The following comments may be offered in conclusion:

1. The approximate technique employed in this paper is an accurate and versatile method. It is felt that a similar approach would be profitable for crack problems in multilayer media.

2. In the case of a thin stiff layer bonded to a soft stratum, the thin layer behaves like a plate. The surface pressure distribution may deviate substantially from the corresponding homogeneous half-space solution.

3. The numerical results contained in this paper were computed for $n = 4$ or 5. It was found that rewriting the base functions into orthogonal forms makes little difference to the final numerical results. In the case of very thin layers, a larger value of n would be necessary and then it may be desirable to rewrite the base functions into orthogonal forms.

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APPENDIX

A composite half-space is composed of two bonded layers which are bonded to a homogeneous half-space. At the surface of the half-space axisymmetric pressure is prescribed within the circle $r \leq a$.

A practical method to compute stresses and displacements within this composite half-space is described in [47]. Suppose the surface tractions are prescribed as follows

$$\sigma_{zz} = \begin{cases} p(r) & r \leq a, \\ 0 & r \leq a. \end{cases} \tag{A.1}$$

$$\sigma_{rz} = \sigma_{\theta z} = 0. \tag{A.2}$$

The vertical displacement $u_z^{(1)}$ within the first layer is given by

$$u_z^{(1)} - u_z^H = \int_0^\infty \bar{u}_n'(\alpha z_1)\, \mathrm{e}^{-4\alpha h_1} \bar{p}(\alpha) J_0(\alpha r)\, \mathrm{d}\alpha, \tag{A.3}$$

where u_z^H is the displacement if the same load is applied to a homogeneous half-space and

$$\bar{p}(\alpha) = \int_0^a p(r) J_0(\alpha r) r\, \mathrm{d}r. \tag{A.4}$$

The term $\bar{u}_n'(\alpha z)$ is

$$2\mu_1 \bar{u}_n'(\alpha z_1) = -[A_1' + (\alpha z_1 + k_1)C_1']\, \mathrm{e}^{-\alpha(z_1 - h_1)} + [B_1' + (\alpha z_1 - k_1)D_1']\, \mathrm{e}^{\alpha(z_1 - h_1)}. \tag{A.5}$$

The functions C' and D' are solved from

$$g_1 C_1' - f_1 D_1' = j_1, \tag{A.6}$$

$$g_2 C_1' + f_2 D_1' = j_2. \tag{A.7}$$

The coefficients, g_1, g_2, f_1, f_2, j_1 and j_2 are given by

$$g_1 = -4\alpha h_1 \left[\frac{\psi(\alpha)}{1 + k_2} - (\beta + \mathrm{e}^{-4\alpha h_2}) \right] + (1 - \mathrm{e}^{-4\alpha h_1}) 4\alpha h_2\, \mathrm{e}^{-4\alpha h_2} + \frac{\mu_2}{\mu_1} \frac{4\alpha h_1 \psi(\alpha)}{1 + k_2} \tag{A.8}$$

$$f_1 = (1-e^{-4\alpha h_1})\left[\frac{\psi(\alpha)}{1+k_2}-(\beta+e^{-4\alpha h_2})\right]-(4\alpha h_1\,e^{-4\alpha h_1})(4\alpha h_2\,e^{-4\alpha h_2}) \qquad (A.9)$$

$$-\frac{\mu_2}{\mu_1}\frac{\psi(\alpha)[1+k_1\,e^{-4\alpha h_1}]}{1+k_2},$$

$$j_1 = -\left[\frac{\psi(\alpha)}{1+k_2}-(\beta+e^{-4\alpha h_2})\right]-(1+4\alpha h_1)4\alpha h_2\,e^{-4\alpha h_2}-\frac{\mu_2}{\mu_1}\frac{k_1\psi(\alpha)}{1+k_2}, \qquad (A.10)$$

$$g_2 = 4\alpha h_1(4\alpha h_2\,e^{-4\alpha h_2})+(1-e^{-4\alpha h_1})\left[\frac{\psi(\alpha)}{1+k_2}-(1+\gamma\,e^{-4\alpha h_2})\,e^{-4\alpha h_2}\right] \qquad (A.11)$$

$$+\frac{\mu_2}{\mu_1}\frac{\psi(\alpha)[k_1+e^{-4\alpha h_1}]}{1+k_2},$$

$$f_2 = -(1-e^{-4\alpha h_1})4\alpha h_2\,e^{-4\alpha h_2}-4\alpha h_1\,e^{-4\alpha h_1}\left[\frac{\psi(\alpha)}{1+k_2}-(1+\gamma\,e^{-4\alpha h_2})\,e^{-4\alpha h_2}\right] \qquad (A.12)$$

$$+\frac{\mu_2}{\mu_1}\frac{\psi(\alpha)4\alpha h_1\,e^{-4\alpha h_1}}{1+k_2},$$

$$j_2 = 4\alpha h_2\,e^{-4\alpha h_2}-(1+4\alpha h_1)\left[\frac{\psi(\alpha)}{1+k_2}-(1+\gamma\,e^{-4\alpha h_2})\,e^{-4\alpha h_2}\right] \qquad (A.13)$$

$$+\frac{\mu_2}{\mu_1}\frac{(1+4\alpha h_1)\psi(\alpha)}{1+k_2}$$

γ and β are two constants defined by

$$\gamma = (k_2-k_3\mu_2/\mu_3)/(1+k_3\mu_2/\mu_3), \qquad (A.14)$$

$$\beta = (k_2+\mu_2/\mu_3)/(1-\mu_2/\mu_3), \qquad (A.15)$$

and $\psi(\alpha)$ is the function

$$\psi(\alpha) \equiv \beta+[\beta\gamma+(4\alpha h_2)^2+1]\,e^{-4\alpha h_2}+\gamma\,e^{-8\alpha h_2}. \qquad (A.16)$$

μ_j and v_j are the shear modulus and Poisson's ratio, respectively, and k_j is equal to (3–4 v_j). The functions A'_1 and B'_1 are evaluated from,

$$A'_1 = -[(2\alpha h_1+k_1)C'_1-D'_1]/2, \qquad (A.17)$$

$$B'_j = -[C'_1+(2\alpha h_1-k_1)D'_1]/2. \qquad (A.18)$$

When there is only one layer over the half-space, and the pressure $p(r)$ is applied over the top surface of medium 2, all the "1" subscripts are replaced by the subscripts 2 in equation (A.5), and the functions A'_2, B'_2, C'_2 and D'_2 are given by

$$C'_2 = (1+4\alpha h_2+\gamma\,e^{-4\alpha h_2})/\psi(\alpha), \qquad (A.19)$$

$$D'_2 = [4\alpha h_2(1+4\alpha h_2)+\gamma\beta+\gamma\,e^{-4\alpha h_2}]/\psi(\alpha), \qquad (A.20)$$

$$A'_2 = -[2\alpha h_2+k_2)C'_2-D'_2]/2, \qquad (A.21)$$

$$B'_2 = -[C'_2+(2\alpha h_2-k_2)D'_2]/2. \qquad (A.22)$$

When the pressure loading is of the form,

$$p(r) = -(1-r^2/a^2)^{n-\frac{1}{2}}, \qquad n = 1, 2, 3 \ldots.$$

the transformed pressure function is

$$\bar{p}(\alpha) = \alpha^{-(2n+1)/2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}.(2n-1)(2n-3)\ldots(3)(1)J_{n+\frac{1}{2}}(\alpha).$$

Also the surface displacement in the corresponding uniform half-space problem is

$$2\mu_1 u_z^H = -(1-\nu)a\frac{\pi(2n-1)(2n-3)\ldots(3)(1)}{2^n n!}\,{}_2F_1\left(\frac{1}{2}, -n; 1; \frac{r^2}{a^2}\right).$$

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Абстракт—На основе классичексой теории упругости решается контактная задача в многочисленной среде. Смешаная краевая задача преобразована в общий приближенный способ, подходящий для расчетов на счетной машине. Указывается на примере задачи Буссинеска, что приближенный метод более точный по сравнению с недавно опубликованным "строгим" анализом. Второй пример, более обыкновенно применимый в инженерной практике, заключается всемя параболический пробойник. Даются численные результаты для двох примеров, для иллюстрации физически важных эффектов мягких и твердых слоев и разных толщин слоев. Затем, анализ обобщается на случай квазистатического удара. Приводятся результаты экспериментов для стального шарика, падающего на нейлонной или резиновый слой. Наблюдается хорошое согласие между результатами аналитическими и экспериментальными.